EULER'S THEOREM: A SIMPLE PROOF

1. Distributive "exhaustion" of the total product

a. In the simple two-factor model all the TP (total product) above the wage bill goes as gain to the fixed factor. Thus the TP is exhausted. The fixed factor gets everything that is left over. This is called "residual" imputation.

b. But what if this residual return to land were different from the MP (marginal product) of land? Then land would be getting more or less than its MP, which would defy the logic of the marginal productivity theory. Fortunately the theorists came up with Euler's Theorem, which shows that this residual must equal the MP of land times the quantity of land.[1]

$$dP/dW \times W + dP/dL \times L = P$$

For formal proof I leave you to any book on micro theory. But an easy proof goes like this. Note what happens if we divide both sides by P. (I'll leave the operation to you). The equation now says that the sum of the left side equals one. The left side now consists of two MP/AP ratios: the first for W (work) and the second for L (land). MP/AP ratios are also "elasticities of production." They are the exponents in the Cobb-Douglas Function, normally denoted as α and β. In fact we could have done it the other way, and derived the theorem from the postulate that $\alpha + \beta$ equal one, then multiplying by P.

Well, that is a comfort, all the pieces fit together. Residual imputation is consistent with marginal productivity imputation, and Euler may be derived from alpha plus beta equal one. We needn't choose one idea over the other, all are consistent.

2. What if returns to scale are other than constant?

Another approach to product imputation is the net product approach, which goes back to the origins of economics with Dr. François Quesnay (ca. 1770), and still has merit today. This approach even lets us cope with conditions where the epsilons add up to other than one -- you must have wondered about that.

[1]This is only true if there are constant returns to scale. (Pedantic theorists, to intimidate their readers, prefer to express this as "assuming homogeneous production functions of degree one," or some such gobbledygook which adds nothing to the subject but sheer terror, in order to endow the pedant with a gloss of intimacy with deep mathematical mysteries.)

If the residual rent is lower than the MP of land then the firm can raise the net gain per acre (rent) by adding land. In the urban land market this phenomenon is called "plottage," the increment in unit land value achieved by assembling small parcels into large ones. More common is "negative plottage" where the MP of land is lower than the residual rent, and large parcels want to be subdivided into smaller ones.

Refer to the paper on the Marginal Product as a Systems Concept - paragraph #4, about adding an 11th acre. That explains that MP = MNP at a point. In class I may go over the net product analysis (time allowing), and show that max ANP is the optimal scale, and at this point ANP = MNP. It follows that MP of land = ANP of land at optimal scale, which is of course also where there are constant returns to scale. Since these are just notes and not a book I merely allude to these things, and indicate how the parts fit into the whole, in case we do cover these matters.

Some operations or improvements to land get stuck back to the left of optimal scale. A water pipeline typically does, for example. The pipe is characterized by "indivisibility," another term for economies of scale, meaning that pipes of larger diameter carry water at lower cost per unit. In these conditions the sum of marginal products must exceed the total product. How to cope? Can life go on?

What happens here is a big topic which has its own branch of theory and practice - it is called "Marginal-cost Pricing." In a nutshell the water service, if optimally priced, would yield a negative rent, usually called a deficit or loss.

Since it is a monopoly, however, it needn't be optimally priced unless intelligently regulated in the public interest, which sometimes happens. One way to cope is to overprice, which is commonplace. Another and better way is to note that the negative rents in the water system are offset by positive rents created in the lands served by the water. The trick is to tap these customer rents to offset the negative rents in the system. This whole matter is much of what public utility economics is all about, and goes by the name of "marginal-cost pricing under conditions of decreasing cost." ("Decreasing cost" is another way of saying "increasing returns to scale.")